Linear: $N_{t+1}(x) = \int K(x-y)\, R\, N_t(y)\, dy$

θ

Question: speed of spread. How to define?

Integrodifference equations in spatial ecology

Lecture 7: Spread in nonlinear equations

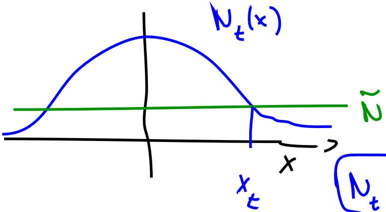
Frithjof Lutscher

$\to$ Gauss $c_g = \sqrt{2\sigma^2 \ln R}$

$\to$ Cauchy $c \to \infty$

Speed: $c = \frac{x_t}{t}$

if $N_0(x) = \delta(x)$



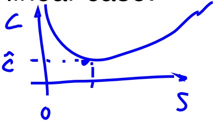
$\boxed{N_t(x_t) = \tilde{N}}$

What has to be different for nonlinear equations?

$N_{t+1}(x) = N_t(x-c)$

$e^{sc} = R\underbrace{\int K(x)e^{sx}\,dx}_{M(s)}$

Recall the minimal speed of a travelling wave in the linear case:



for linear

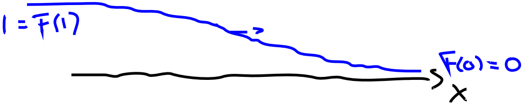
$$\boxed{\hat{c} = \min_{s>0} \frac{1}{s} \ln(RM(s)).}$$

Nonlinear:

- No explicit solutions (as in the linear case)
- Potentially slower speed since lower density behind front.
- Set-up: $F(0) = 0$ and $F(1) = 1$
- Guess the shape

$N_{t+1}(x) = \int K(x-y)F(N_t(y))\,dy$

no Allee effect,



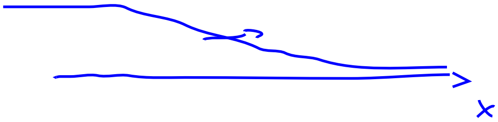
The travelling wave approach

$N_t(x-c)$ $\qquad\qquad\qquad\qquad$ $N_t(y)$

$$N^*(x-c) = Q[N^*](x) = \int_{-\infty}^{\infty} K(x-y)F(N^*(y))dy$$

Limits as $|x| \to \infty$.

$N^*(x) \to 1$ as $x \to -\infty$ $\qquad$ $N^*(x) \to 0$ as $x \to +\infty$



$N_{t+1}(x) = N_t(x-c)$

1) Existence?

2) Speed c?

Boundedness by the linearized speed

Lemma:
Assume that the moment generating function of K exists in some neighborhood of zero and that the growth function satisfies $F(N) \leq F'(0)N$. Then the minimum traveling front speed of the nonlinear equation is bounded above by $\hat{c}$ in the linear equation where $R = F'(0)$.

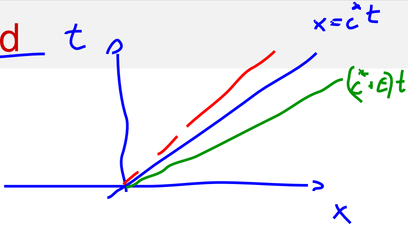
?

$$N^*(x-c) = \int K(x-y) F(N^*(y))\,dy \leq F'(0) \int K(x-y) N^*(y)\,dy$$

Take exponential transform: $\int N^*(x) e^{sx}\,dx$

$$e^{sc} \leq F'(0) \int K(x) e^{sx}\,dx = F'(0)\, M(s)$$

The asymptotic spreading speed



Aronson & Weinberger '78

Weinberger '82

$\frac{|x|}{t} > c^* + \epsilon$

Definition:
The number c^* is called the asymptotic spreading speed if it satisfies the conditions

$$\limsup_{t\to\infty} \max_{|x|>(c^*+\epsilon)t} N_t(x) = 0, \qquad \liminf_{t\to\infty} \min_{|x|<(c^*-\epsilon)t} N_t(x) \geq \beta > 0,$$

$\frac{|x|}{t} < c^* - \epsilon$

for all $\epsilon > 0$ and some $\beta > 0$, and N_0 is compactly supported and N_t is defined by the iteration $N_{t+1} = Q[N_t]$.

observe with speed faster than c^*

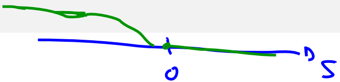
slower

The operator Q

cont. functions with values in $[0,1]$

i) (Translation invariance) $Q[N(\cdot - a)](x) = Q[N](x-a)$ for all $a \in \mathbb{R}$.
ii) (Invariance of $C_{[0,1]}$) $N \in C_{[0,1]} \Rightarrow Q[N] \in C_{[0,1]}$.
iii) (Fixed points) $Q[0] = 0$, $Q[1] = 1$, $Q[a] > a$ for $a \in (0,1)$.
iv) (Monotonicity) $0 \leq N(x) \leq M(x) \leq 1 \Rightarrow Q[N](x) \leq Q[M](x)$
v) (Continuity) If $\{f_j\} \subset C_{[0,1]}$ and $f_j \to f$ uniformly on compact subsets of $\mathbb{R}$ then $Q[f_j] \to Q[f]$ pointwise as $j \to \infty$.
vi) (Compactness) Every sequence $\{f_j\}$ in $C_{[0,1]}$ has a subsequence $\{f_{j_i}\}$ such that $\{Q[f_{j_i}]\}$ converges uniformly on every bounded subset of $\mathbb{R}$.

An abstract theorem (Weinberger '82)



Assume that Q satisfies i)–v). Then there exits an asymptotic spreading speed $c^* > 0$ for Q. If, in addition, vi) holds, then for every $c \geq c^*$ there exists a continuous monotone traveling wave solution $N_t(x) = W(x - ct)$ of Q with $W(\infty) = 0$ and $W(-\infty) = 1$. No such traveling wave exists for $c < c^*$. How to calculate c^*?

Define c^*: Pick $\phi : \mathbb{R} \to [0,1]$, non-increasing, $\phi(s) = 0$ for $s \geq 0$, $\phi(-\infty) = 1$

Define: $a_0(c,s) = \phi(s)$

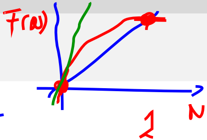
$$a_{k+1}(c,s) = \max\{\phi(s),\ Q[a_k(c, \cdot + s + c)](0)\}$$

$$Q[a_k(c, \cdot + c)](s)$$

show $a_k \to a$

$$c^* = \sup\{c \mid a(c,\infty) = 1\}$$

Theorem in the monotone case



$Q[N] = \int K(x-y) F(N(y))\, dy$

Assume that

(F1): $F(0) = 0$, $F(1) = 1$ are the only two fixed points of F.

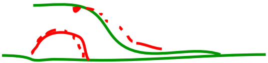
(F2): Fixed point $N = 0$ is unstable and $N = 1$ is stable.

(F3): F is continuous and monotone increasing.

(F4): F is differentiable at 0 and linearly bounded, i.e. $F(N) \leq F'(0)N$.

(K1): K is continuous, symmetric and compactly supported.

Then there exists an asymptotic spreading speed, c^*, for Q. Furthermore, c^* equals the minimal traveling wave speed $\hat{c}$, where $R = F'(0)$. Finally, for all $c \geq c^*$ there exists a monotone traveling front solution, connecting 0 and 1 with speed c, but for $c < c^*$ no such traveling front solution exists.



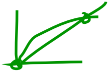
Proof

Translation invariance: $Q[N(\cdot - a)](x) = \underline{Q[N](x-a)}$

$$\int \underbrace{K(x-a-y)}F(N(y))\,dy = \int K(x-(y+a))F(N(y))\,dy = \int K(x-z)F(N(z-a))\,dz$$

$K(x-y)$, $K(x,y)$ $\qquad z = y+a$

Invariance of $C[0,1]$: Assume $0 \le N(x) \le 1$. Then $0 \le F(N(x)) \le 1$



$$0 \le \int K(x-y)F(N(y))\,dy \le \int K(x-y)\,dy = 1$$

since K is a PDF

Fixed points: $F(0) = 0$, $F(1) = 1$

$\forall x$

$$\int K(x-y)F(0)\,dy = 0, \quad \int K(x-y)F(1)\,dy = \int K(x-y)\,dy = 1$$

Proof

$Q[a] > a$ for $0 < a < 1$. Show: $F(a) > a$ ✓

Proof Continuity:

Let L be the Lipschitz const. of F

Let $\{f_j\}$ be a sequence in $C[0,1]$, $f_j \to f$ unif. on cp. sets.

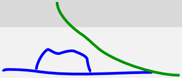
$$\left|Q[f_j](x) - Q[f](x)\right| \le \left|\int K(x-y)\big(F(f_j(y)) - F(f(y))\big)\,dy\right|$$

$$\le \int_{|y|>m} K(x-y)\underbrace{\left|F(f_j(y)) - F(f(y))\right|}_{\le 2}dy + \int_{|y|\le m} K(x-y)\left|F(f_j(y)) - F(f(y))\right|dy$$

$$\le 2\underbrace{\int_{|y|>m} K(x-y)\,dy}_{<\varepsilon} + L\int_{|y|\le m} K(x-y)\underbrace{|f_j(y)-f(y)|}_{<\varepsilon}dy < C\cdot\varepsilon$$

Proof

Lemma: $c^* \le \hat{c}$



N_0 compact support $\Rightarrow$ $N_0(x) \le A e^{-sx}$ $\wedge$ $\forall s>0\ \exists A$:

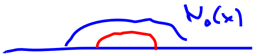
$$N_{t+1}(x) = \int K(x-y)F(N_t(y))\,dy \le F'(0)\int K(x-y)N_t(y)\,dy$$

-> solution of nonlin eq. is bounded by sol. of lin eq.

-> for lin. eq. min speed $\hat{c}$, choose s that corresponds to $\hat{c}$ to bound N_0

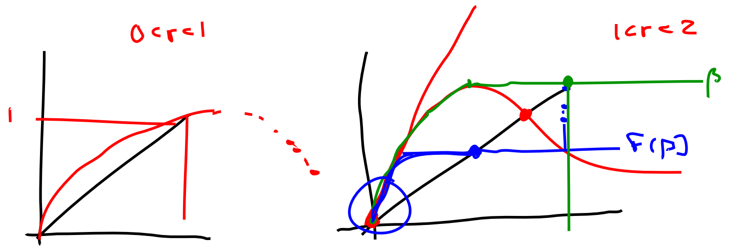
$\square$

True $c^* \ge \hat{c}$: Idea: subsolution:



 $F^{+'}(0) = F'(0) = F^{-'}(0)$

$F(N) = N\exp(r(1-N))$ is monotone for $0 < r < 1$ but not for $r > 1$.
Existence of a spreading speed via upper and lower bounds.



Define F^+, monotone $C[0,\beta]$

$\Downarrow$

spreading speed c_+^*

not monotone

$Q[a]>a$ ✓

invariance of $C[0,1]$ ↯

order-preserving ↯

Define F^- monotone $C[0,F(\beta)] \to c_-^*$

$F^-(N) \le F(N) \le F^+(N)$

$Q^-[N] \le Q[N] \le Q^+[N]$

$c_-^* \le c^* \le c_+^*$

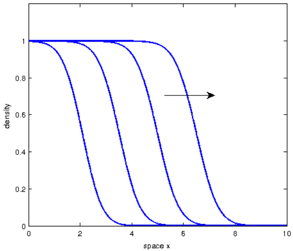
Theorem in the non-monotone case

Assume that the conditions of the previous Theorem hold except for the monotonicity condition. Instead, assume that F has a unique maximum $\hat{F} = F(\hat{N})$ for some $0 < \hat{N} < 1$. Assume that F is increasing on $(0, \hat{N})$ and decreasing on $(\hat{N}, \hat{F})$. Define functions $F^{\pm}$ and operators $Q^{\pm}$. Then the following hold.

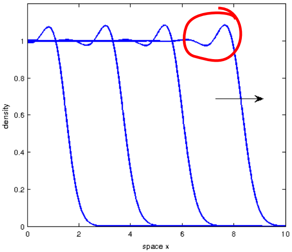
1. $F^{\pm}$ satisfy conditions (F1)–(F4).
2. $Q^{\pm}$ satisfy the assumptions of monotone Theorem; their spreading speeds $c^*_{\pm}$ exist.
3. Q has a spreading speed c^* and $c^*_- = c^* = c^*_+$.

Illustration

$0 < r < 1$



$1 < r < 2$



non-monotone TW

What happens when $r > 2$?