Integrodifference equations in spatial ecology

Lecture 6: Spread in linear equations

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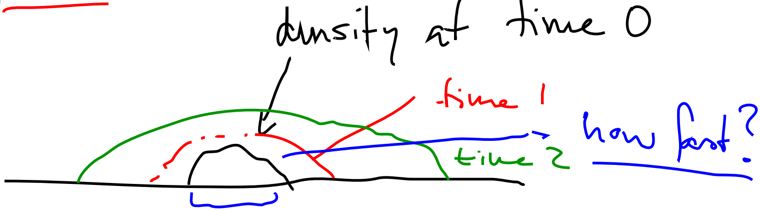
The importance of spread rates $\frac{\partial}{\partial t} u = f(u) + D\frac{\partial^2}{\partial x^2}u$

Fisher eq. 1937

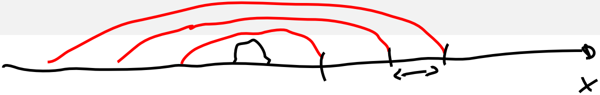
rate of spread $c = 2\sqrt{Df'(0)}$? ?

- Spatial spread follows local establishment.
- Biological invasions.
- For IDEs in particular: How to spread rates depend on dispersal patterns?



Emerald ash borer ~ 2000 near Detroit

The set-up



- infinite landscape
- homogeneous
- no Allee effect

$K(x,y) = \tilde{K}(x-y)$

$$\boxed{N_{t+1}(x) = \int_{-\infty}^{\infty} K(x-y)F(N_t(y))dy = (K * F(N_t))(x).}$$

convolution

Typical starting point: linear $F(N) = RN$.

How to measure spread?

$$K(x-y) = \text{Gaussian}(x-y; \mu, \sigma^2)$$
$$>0$$

No sharp edge

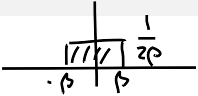
$N_0 > 0$ for $[-\delta, \delta]$

$$N_1(x) = \int K(x-y) R N_0(y)\, dy > 0 \quad \text{for all } x.$$

Cannot use support of N_t to measure spread.

The point-release experiment I

- $N_0(x) = \nu\delta(x)$ n Dirac



let $\beta \to 0$

distribution

- Gaussian kernel (mean zero, variance σ^2)

$G(x; \mu=0, \sigma^2)$

$$N_1(x) = R\int_{-\infty}^{\infty} G(x-y;\sigma^2)\,\delta(y)\,dy = R\,G(x;\sigma^2)$$

$$N_2(x) = R^2\int_{-\infty}^{\infty} G(x-y;\sigma^2)\,G(y)\,dy = R^2\,G(x;2\sigma^2)$$

$\vdots$

$$N_t(x) = \dots = R^t\,G(x;t\cdot\sigma^2)$$

$$\boxed{N_t(x) = \frac{R^t}{\sqrt{2\pi t\sigma^2}}\,e^{-\frac{x^2}{2t\sigma^2}}}$$

The point-release experiment II

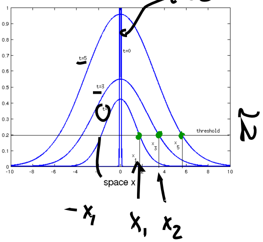
1) Speed depends on time

2) $\frac{x_t}{t} \to \sqrt{2\sigma^2 \ln R}$ as $t \to \infty$

$$\frac{x_t}{t} = \sqrt{2\sigma^2 \ln R - \underbrace{\frac{2\sigma^2 \ln(\sqrt{\cdot t}\tilde{N})}{t}}_{0}}$$

- Detection threshold $\tilde{N}$.

Biology:



$N_t(x_t) = \tilde{N}$ Definition of $x_t > 0$

Idea: Find $x_t - x_{t-1}$: distance per generation

$\frac{x_t}{t}$ speed since introduction

$$\tilde{N} = \frac{R^t}{\sqrt{2\pi t\sigma^2}} e^{-\frac{x_t^2}{2t\sigma^2}}$$

$$e^{\frac{x_t^2}{2t\sigma^2}} = \frac{R^t}{\sqrt{2\pi t\sigma^2}\,\tilde{N}}$$

$$x_t = \sqrt{2\sigma^2 t^2 \ln(R) - 2\sigma^2 t \ln(\cdots)}$$

$$R^t e^{-\frac{x_t^2}{2t\sigma^2}} = \sqrt{2\pi t\sigma^2}\,\tilde{N}$$

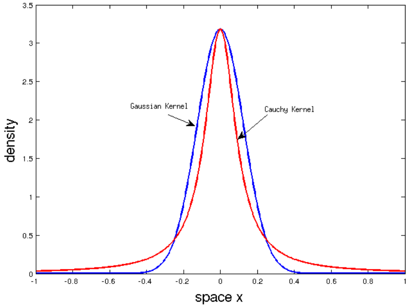
$$\frac{x_t^2}{2t\sigma^2} = t\ln R - \ln\left(\sqrt{2\pi t\sigma^2}\,\tilde{N}\right)$$

Asymptotic speed

$c = \sqrt{2\sigma^2 \ln(R)}$ for Gaussian kernel

The point-release experiment III

- $N_0(x) = \delta(x)$
- Cauchy kernel $K(x) = \frac{1}{\pi}\frac{\beta}{\beta^2+x^2}$



The point-release experiment IV

Fourier transform (scaled)

$$\widehat{N}(\omega) = \sqrt{\int_{-\infty}^{\infty} N(x)e^{ix\omega}dx}$$

For the Cauchy kernel

$$\boxed{\widehat{K}(\omega) = e^{\beta|\omega|}}$$

$$\widehat{N*K}(\omega) = \hat{N}(\omega)\cdot\hat{K}(\omega)$$

$$N_{t+1} = R(K*N_t)$$

$$\hat{N}_{t+1}(\omega) = R\hat{K}\cdot\hat{N}_t(\omega)$$

$$N_0 = \delta \rightarrow \hat{N}_0 = 1$$

$$\hat{N}_t(\omega) = R^t\hat{K}^t(\omega)$$

inverse transform?

$$\hat{K}^t(\omega) = e^{t\beta|\omega|}$$

$$\Rightarrow \boxed{N_t(x) = \frac{R^t}{\pi}\cdot\frac{\beta t}{\beta^2 t^2 + x^2}}$$

$\tilde{N} = N_t(x_t)$ solve for x_t

$$\frac{x_t}{t} = \sqrt{\frac{R^t\beta\cancel{t}}{\tilde{N}\pi t} - \beta^2\cancel{t^2}} \sim \sqrt{\frac{R^t\beta}{\tilde{N}\pi t}} \to \infty$$

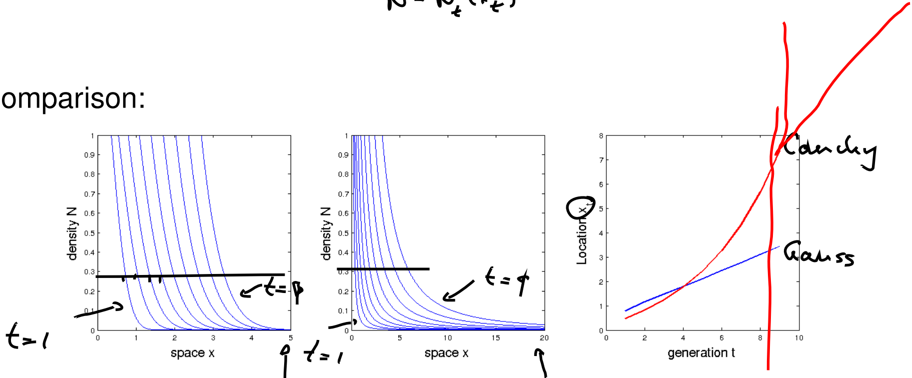
exp in t

lin t

The point-release experiment V

$$\tilde{N} = N_t(x_t)$$

Comparison:



Gaussian

Cauchy

accelerating invasion

Kot, Ecology 1996

Alternative approach

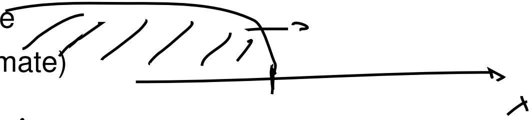
The area release experiment

idea

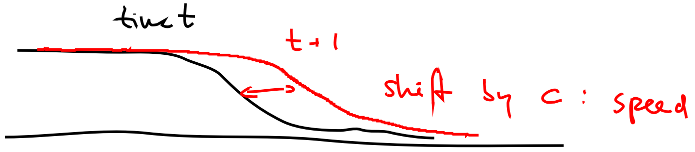
$\boxed{N_t(x_t) = \tilde{N}}$

requires knowledge of N_t

- Population established over a large range
- continues to expand
- long after point release
- new opportunities (climate)



Special kind of solution



Travelling profile, exponential ansatz and moment generating function

e^{-sx}

x

Travelling profile, speed c:

$N_{t+1} = R(K * N_t)$

$$N_{t+1}(x) = N_t(x-c) = N^*(x-c)$$

profile

$$N^*(x-c) = R\int K(x-y)\,N^*(y)\,dy$$

linear eq.

$\boxed{N^* \sim e^{-sx}}$

$$e^{-s(x-c)} = R\int K(x-y)\,e^{-sy}\,dy = R\int K(y)\,e^{-s(x-y)}\,dy$$

$$e^{sc} = R\boxed{\int K(y)\,e^{sy}\,dy} = R\cdot M(s)$$

moment generating function of K

$c = \frac{1}{s}\ln(R\,M(s))$

dispersion relation

The minimal travelling wave speed

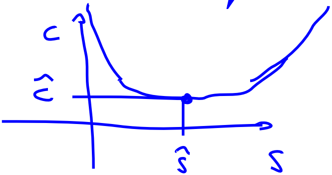
The dispersion relation

$$\boxed{c = c(s) = \frac{1}{s}\ln(RM(s)),} \qquad \underline{s > 0}$$

if $s \to 0 \Rightarrow$ "flat" wave $\frac{1}{s} \to \infty \Rightarrow c \to \infty$ fast

if $s \to \infty \Rightarrow$ "steep" wave $M(s) \to \infty$ "fast" $\Rightarrow c \to \infty$ fast

in general $c(s)$ is concave up



The minimal travelling wave speed

$$\hat{c} = \min_{s>0} \frac{1}{s}\ln(RM(s)).$$

Calculations for Gauss

$$\hat{c} = \min_{s>0} \frac{1}{s} \ln(R\,M(s))$$

$$K(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}, \quad M(s) = \exp\left(\frac{\sigma^2 s^2}{2}\right)$$

$$c(s) = \frac{1}{s}\ln R + \frac{1}{s}\frac{\sigma^2 s^2}{2} \qquad c'(s) = -\frac{\ln R}{s^2} + \frac{\sigma^2}{2} \overset{!}{=} 0$$

$$\frac{\ln R}{s^2} = \frac{\sigma^2}{2} \quad \text{or} \quad s^2 = \frac{2\ln R}{\sigma^2} \qquad \hat{s} = \sqrt{\frac{2\ln R}{\sigma^2}}$$

$$\hat{c} = c(\hat{s}) = \sqrt{\frac{\sigma^2}{2\ln R}}\cdot \ln R + \frac{\sigma^2}{2}\sqrt{\frac{2\ln R}{\sigma^2}} = \sqrt{\frac{\sigma^2}{2}\ln R} + \sqrt{\frac{\sigma^2}{2}\ln R}$$

$$\boxed{\hat{c} = \sqrt{2\sigma^2 \ln R}}$$ minimal TW speed $= c$ asymptotic

Parametric representation

dispersion relation

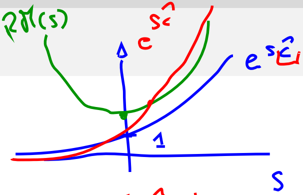
From

$$e^{s\hat{c}} = RM(s)$$

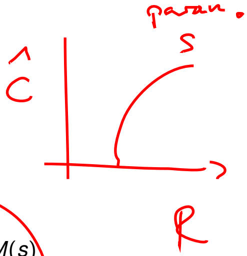
plus tangency condition

$$\hat{c}\, e^{s\hat{c}} = RM'(s)$$

divide by



minimal $\hat c$: tangent



obtain

$$\hat{c} = \frac{M'(s)}{M(s)}, \qquad R = \frac{e^{sM'(s)/M(s)}}{M(s)}$$

The Laplace kernel

$$K(x) = \frac{a}{2}e^{-a|x|}, \quad M(s) = \frac{a^2}{a^2 - s^2}, \qquad |s| < a$$

Parametric representation

$$\hat{c} = \frac{2s}{a^2 - s^2}, \quad R = \left(1 - \frac{s^2}{a^2}\right)\exp\left(\frac{2s^2}{a^2 - s^2}\right).$$

Set $\tilde{s} = \frac{2s^2}{a^2 - s^2}$, to get

$$R = \frac{2e^{\tilde{s}}}{2 + \tilde{s}} \quad \text{and} \quad a\hat{c} = \frac{2sa}{a^2 - s^2} = \frac{\tilde{s}a}{s} = \sqrt{\tilde{s}^2 + 2\tilde{s}}.$$

Compare for three kernels