Integrodifference equations in spatial ecology

Lecture 16: Pattern formation II

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Explicit example

$$N_t e^{r(1-N_t-P_t)}$$

$$N_{t+1} = \cancel{N_t e^{r(1-N_t)} e^{-\rho P_t}}, \qquad P_{t+1} = N_t \left(1 - e^{\rho P_t}\right)$$

with steady states

$$N = 1/\rho, \qquad P = 1 - 1/\rho$$

and Jacobi matrix

$$< 1$$

$$J = \begin{bmatrix} 1 - r/\rho & -r/\rho \\ \rho - 1 & \underbrace{1}_{=1} \end{bmatrix}.$$

if $\hat{k}_N, \hat{k}_P \geq 0$

condition for t.-b.f.: $(a_{11}-1)(a_{22}-1) < 0$ not possible
no activator

Kernels and transforms

Laplace:

$$K_L(x) = \frac{a}{2} e^{-a|x|} \qquad \text{with} \qquad \widehat{K}(\omega) = \frac{a^2}{a^2+\omega^2}$$

Double Gamma:

$$K_{DG}(x) = \frac{b^2}{2}|x|e^{-b|x|} \qquad \text{with} \qquad \widehat{K}(\omega) = \frac{b^2(b^2-\omega^2)}{(b^2+\omega^2)^2} \quad < 0 \,!$$

$|b| < |\omega|$

For a ‘plus-one’ bifurcation, we want $K_N = K_{DG}$ and $K_P = K_L$.

The dispersion relation

$1 - \hat{k}_N a_{11} - \hat{k}_p a_{22} + \hat{k}_N \hat{k}_p \det J \overset{?}{<} 0$

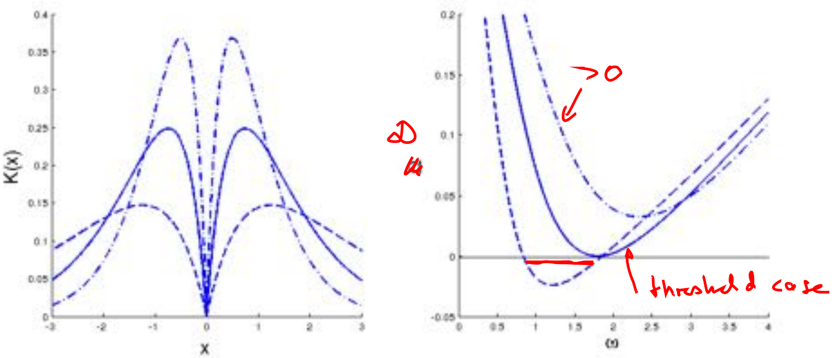
$$\mathcal{D}(\omega) := 1 - \widehat{K}_N \left(1 - \underline{\frac{r}{\rho}}\right) - \widehat{K}_P + \widehat{K}_N \widehat{K}_P \left(1 + r - \frac{2r}{\rho}\right)$$

Want: $\mathcal{D} < 0$

$$1 - \frac{b^2(b^2 - \omega^2)}{(b^2 + \omega^2)^2} \left(1 - \frac{r}{\rho}\right) - \frac{a^2}{a^2 + \omega^2} + \frac{b^2(b^2 - \omega^2)}{(b^2 + \omega^2)^2} \frac{a^2}{a^2 + \omega^2} \left(1 + r - \frac{2r}{\rho}\right) < 0.$$

Dispersion relation

Double Gamma



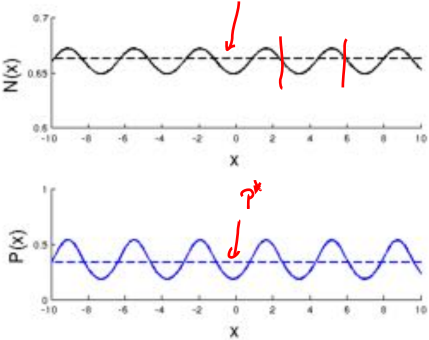
fix r, g, a
vary b

Patterns

$b = 1.3$

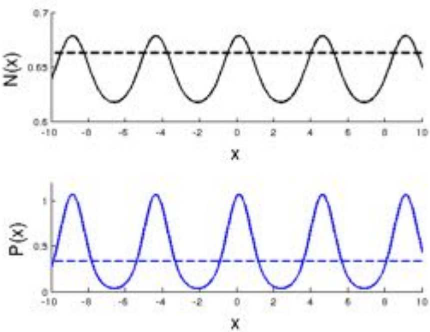
N^*

P^*



6000 time steps

$b = 1.0$

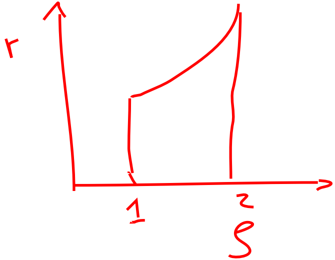


More calculations

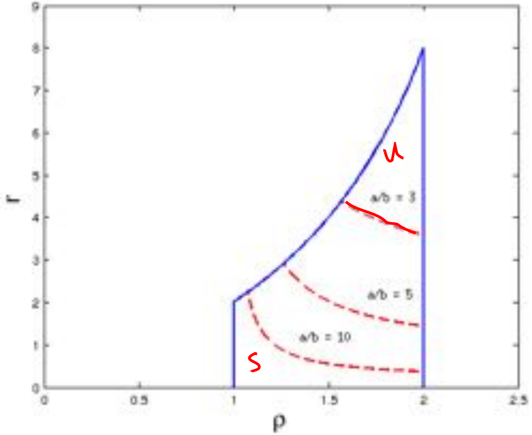
write: $\rho = 1 + y = 1 + \frac{1}{a^2} \frac{u(u+1)(3-u)}{2(u^2-3)}$

$r = x \cdot \rho = \frac{2u(u^2-3)}{(1-u)^2} \rho$

parametric curves



The onset of patterns



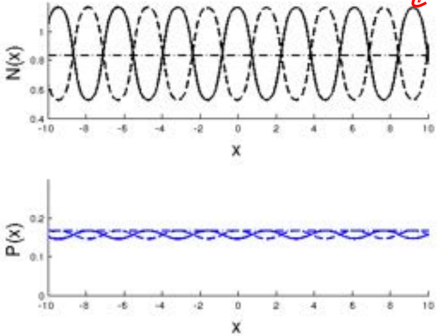
$$K_p = \frac{a}{2} e^{-a|x|}$$

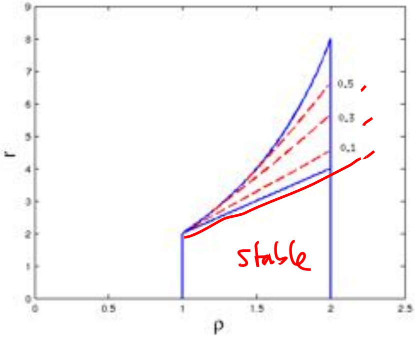
$$K_N = \frac{b^2}{2} |x| e^{-b|x|}$$

A ‘minus-one’ bifurcation

$t = 0$ mod 2 - - - -

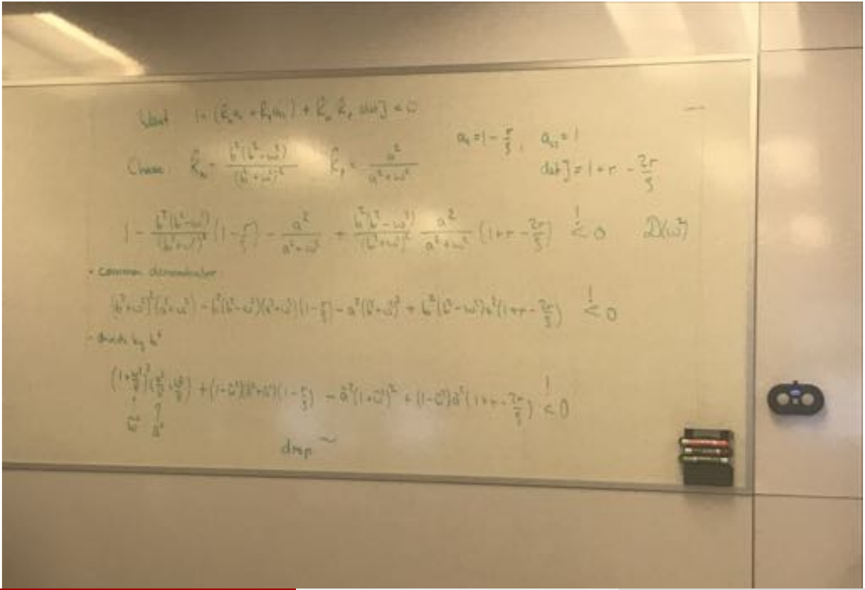
$t = 1$ mod 2 ———



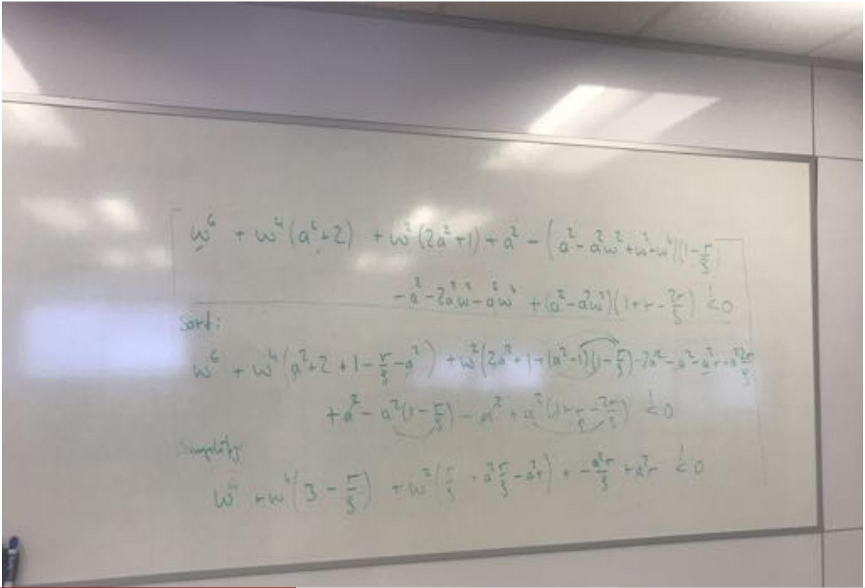


stable

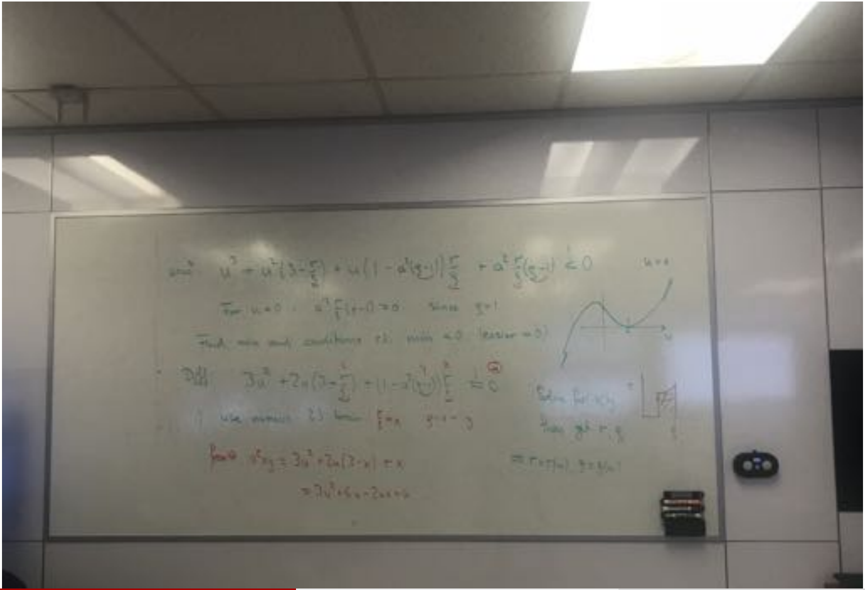
Whiteboard panel 1



Whiteboard panel 2



Whiteboard panel 3



Whiteboard panel 4