

Integrodifference equations in spatial ecology

Lecture 16: Pattern formation II

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Explicit example

$$N_{t+1} = \overbrace{N_t e^{r(1-N_t-P_t)}}^{N_t e^{r(1-N_t-P_t)}}, \quad P_{t+1} = N_t (1 - e^{\rho P_t})$$

with steady states

$$N = 1/\rho, \quad P = 1 - 1/\rho$$

and Jacobi matrix

$$J = \begin{bmatrix} 1 - r/\rho & -r/\rho \\ \rho - 1 & 1 \end{bmatrix}.$$

< 1

$$\text{if } \hat{K}_N, \hat{K}_P \geq 0$$

condition for +1 - bif: $(a_{11} - 1)(a_{22} - 1) < 0$ not possible
no activator

Kernels and transforms

Laplace:

$$K_L(x) = \frac{a}{2} e^{-a|x|} \quad \text{with} \quad \hat{K}(\omega) = \frac{a^2}{a^2 + \omega^2} \quad \checkmark$$

Double Gamma:

$$K_{DG}(x) = \frac{b^2}{2} |x| e^{-b|x|} \quad \text{with} \quad \hat{K}(\omega) = \frac{b^2 (b^2 - \omega^2)}{(b^2 + \omega^2)^2} \quad \begin{array}{l} < 0 \uparrow \\ |b| < |\omega| \end{array}$$

For a 'plus-one' bifurcation, we want $K_N = K_{DG}$ and $K_P = K_L$.

The dispersion relation

$$1 - \hat{K}_N a_{11} - \hat{K}_P a_{22} + \hat{K}_N \hat{K}_P \text{ det} \stackrel{!}{<} 0$$

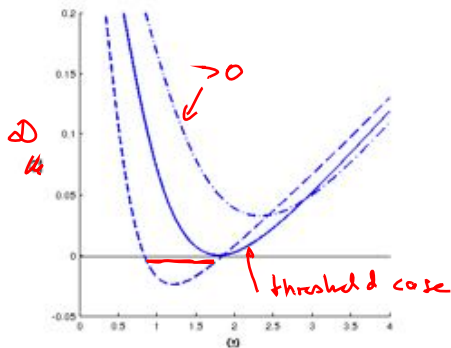
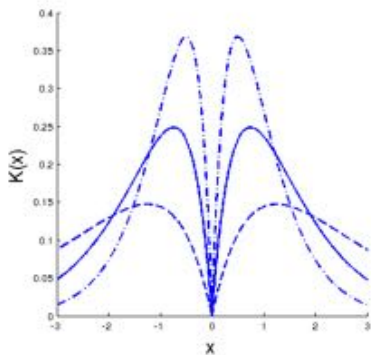
$$D(\omega) := 1 - \hat{K}_N \left(1 - \frac{r}{\rho}\right) - \hat{K}_P + \hat{K}_N \hat{K}_P \left(1 + r - \frac{2r}{\rho}\right)$$

Want: $D < 0$

$$1 - \frac{b^2(b^2 - \omega^2)}{(b^2 + \omega^2)^2} \left(1 - \frac{r}{\rho}\right) - \frac{a^2}{a^2 + \omega^2} + \frac{b^2(b^2 - \omega^2)}{(b^2 + \omega^2)^2} \frac{a^2}{a^2 + \omega^2} \left(1 + r - \frac{2r}{\rho}\right) < 0.$$

Dispersion relation

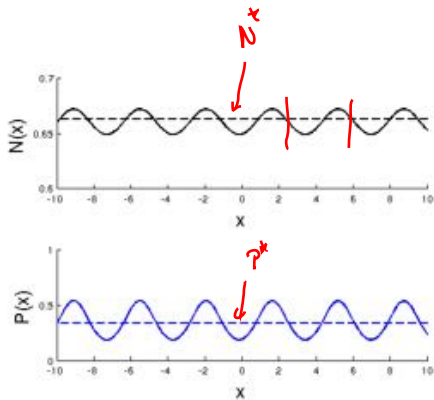
Double Gamma α



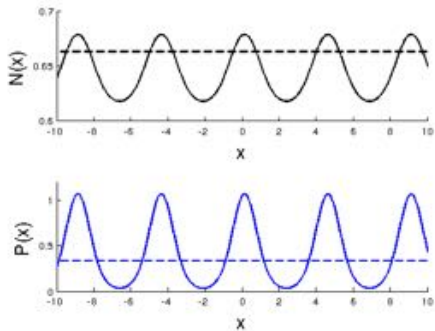
fix r, g, a
vary b

Patterns

$b = 1.3$



$b = 1.0$



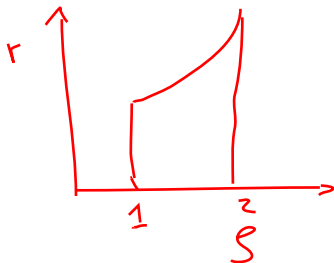
6000 time steps

More calculations

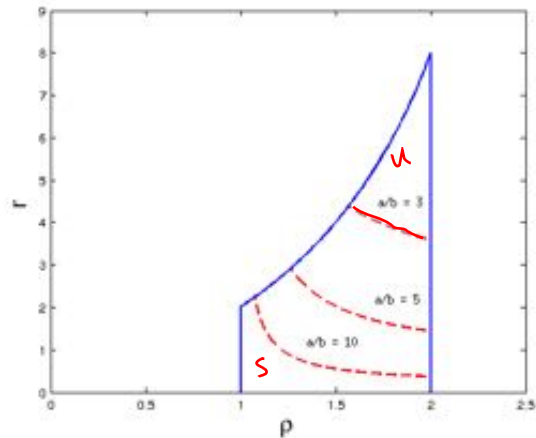
$$\text{witz: } g = 1 + y = 1 + \frac{1}{a^2} \frac{u(u+1)(3-u)}{2(u^2-3)}$$

$$r = x \cdot g = \frac{2u(u^2-3)}{(1-u)^2} \mathcal{P}$$

parametric curves



The onset of patterns

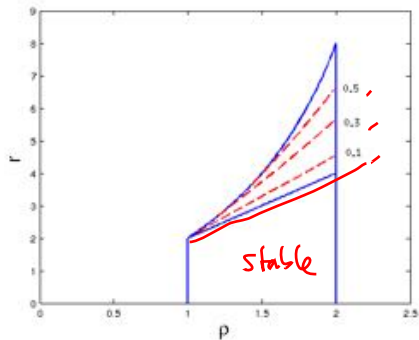
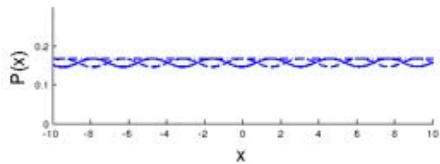
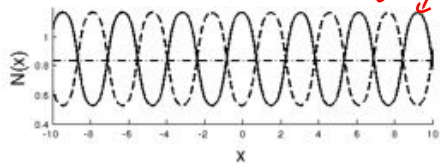


$$K_T = \frac{a}{2} e^{-a|x|}$$

$$K_N = \frac{B^2}{2} |x| e^{-\zeta|x|}$$

A 'minus-one' bifurcation

$t = 0 \text{ mod } 2$
 $t = 1 \text{ mod } 2$



Whiteboard panel 1

$$\text{Want } 1 - (k_1 a + k_2 a_1) + k_2 z_1 \omega^2 = 0$$

$$\text{Choose } k_1 = \frac{b^2(b^2 - \omega^2)}{(b^2 + \omega^2)^2} \quad k_2 = \frac{a^2}{a^2 + \omega^2} \quad a_1 = 1 - \frac{r}{s}, \quad a_2 = 1$$

$$\det J = 1 + r - \frac{2r}{s}$$

$$1 - \frac{b^2(b^2 - \omega^2)}{(b^2 + \omega^2)^2} \left(1 - \frac{r}{s}\right) - \frac{a^2}{a^2 + \omega^2} + \frac{b^2(b^2 - \omega^2)}{(b^2 + \omega^2)^2} \frac{a^2}{a^2 + \omega^2} \left(1 + r - \frac{2r}{s}\right) \stackrel{!}{\leq} 0 \quad 2k(\omega^2)$$

- common denominator

$$(b^2 + \omega^2)^2 (a^2 + \omega^2) - b^2(b^2 - \omega^2)(a^2 + \omega^2) \left(1 - \frac{r}{s}\right) - a^2(b^2 + \omega^2)^2 + b^2(b^2 - \omega^2)^2 \left(1 + r - \frac{2r}{s}\right) \stackrel{!}{\leq} 0$$

- divide by b^4

$$\left(1 + \frac{\omega^2}{b^2}\right)^2 \left(\frac{a^2}{b^2} + \frac{\omega^2}{b^2}\right) - \frac{b^2 - \omega^2}{b^2} \left(\frac{a^2}{b^2} + \frac{\omega^2}{b^2}\right) \left(1 - \frac{r}{s}\right) - \left(\frac{a^2}{b^2} + \frac{\omega^2}{b^2}\right)^2 + \left(\frac{b^2 - \omega^2}{b^2}\right)^2 \left(1 + r - \frac{2r}{s}\right) \stackrel{!}{\leq} 0$$

drop ~

Whiteboard panel 2

$$\omega^6 + \omega^4(a^2+2) + \omega^3(2a^2+1) + a^2 - \left(\frac{a^2 - a^2\omega^2 + \omega^2 - \omega^4}{1 - \omega^2} \right) - \frac{-a^2 - 2a^2\omega^2 - a^2\omega^4 + (a^2 - a^2\omega^2)(1 + \omega - \frac{2\omega^2}{3})}{3} \leq 0$$

Sort:

$$\omega^6 + \omega^4 \left(a^2 + 2 + 1 - \frac{\omega^2}{3} - a^2 \right) + \omega^2 \left(2a^2 + 1 + (a^2 - 1) \left(1 - \frac{\omega^2}{3} \right) - \frac{2a^2}{3} - a^2 \frac{\omega^2}{3} - a^2 \frac{2\omega^2}{3} \right) + a^2 - a^2 \left(1 - \frac{\omega^2}{3} \right) - a^2 + a^2 \left(1 + \omega - \frac{2\omega^2}{3} \right) \leq 0$$

Simplify:

$$\omega^6 + \omega^4 \left(3 - \frac{\omega^2}{3} \right) + \omega^2 \left(\frac{\omega^2}{3} + a^2 \frac{\omega^2}{3} - a^2 \right) + \frac{-a^2\omega}{3} + a^2 - \leq 0$$

Whiteboard panel 3

$$\text{mit } u^3 + u^2(3-\frac{1}{y}) + u(1-\frac{1}{y})\frac{1}{y} + \frac{1}{y^2}(y-1) \leq 0 \quad u=0$$

$$\text{Für } u=0: u^2 f(y) = 0 \Rightarrow \text{wenn } y=1$$

Für $u > 0$ sind conditions z.B. mit $u < 0$ (lassen $= 0$)

$$\text{Daher } 3u^3 + 2u(3-\frac{1}{y}) + (1-\frac{1}{y})\frac{1}{y} \stackrel{!}{=} 0$$

via ansatz z.B. für $y=1 \Rightarrow 0$

$$\begin{aligned} \text{für } u \neq 0: 3u^2 + 2u(3-\frac{1}{y}) + \frac{1}{y} &= 0 \\ &= 3u^2 + 6u - 2u + \frac{1}{y} \end{aligned}$$



Skizze für $u > 0$
für $y < 1$

\Rightarrow minimal $y > 1$

Whiteboard panel 4

$$u^3 + 3u^2 - u^2 x + 4x - u(3u^2 + 6u - 2ux + x) + 3u^2 + 6u - 2ux + x = 0$$

linear in x:

$$x = \frac{2u^3 - 6u}{u^2 - 2u + 1} = \frac{2u(u^2 - 3)}{(u-1)^2}$$

Now:

$$a^2 y = \frac{3u^2 + 6u}{x} - 2u + 1$$

$$= \frac{(u-1)^2 (3u^2 + 6u)}{2u(u^2 - 3)} - 2u + 1$$

$$= \frac{1}{2(u^2 - 3)} \left[(u-1)^2 3(u+2) - (2u-1)(u^2 - 3) \right] = \frac{u}{2(u^2 - 3)} (u+1)(3-u)$$