Integrodifference equations in spatial ecology

Lecture 14: Interacting species

Frithjof Lutscher

Three common types of interactions

- predation
- competition
- mutualism

→ Homework

Non-spatial model for interaction

prey/host predator/parasitoid

$$N_{t+1} = F(N_t, P_t), \qquad P_{t+1} = G(N_t, P_t).$$

Properties of F and G

competition: if $P \nearrow$ then $F \searrow$ $\frac{\partial F}{\partial P} < 0$ and $\frac{\partial G}{\partial N} < 0$

predation of N by P: $\frac{\partial F}{\partial P} < 0$, $\frac{\partial G}{\partial N} > 0$

Steady states and stability

$$N^* = F(N^*, P^*), \qquad P^* = G(N^*, P^*)$$

Linearized stability:

$$J = \begin{bmatrix} \frac{\partial F}{\partial N} & \frac{\partial F}{\partial P} \\ \frac{\partial G}{\partial N} & \frac{\partial G}{\partial P} \end{bmatrix}\Bigg|_{(N^*,P^*)}$$

Linear discrete eq.

$$\begin{pmatrix} n \\ p \end{pmatrix}_{t+1} = J \begin{pmatrix} n \\ p \end{pmatrix}_t$$

$$\Downarrow$$

$$\begin{pmatrix} n \\ p \end{pmatrix}_t = J^t \begin{pmatrix} n \\ p \end{pmatrix}_0$$

$\Rightarrow$ powers of J. $\Rightarrow$ study eigenvalues of J.

If $|\lambda_{1,2}| < 1$

$\Downarrow$

stability

If $|\lambda_1| > 1$ or $|\lambda_2| > 1$

$\Downarrow$

instability

$$\lambda_{1/2} = \frac{1}{2}\left(tr \pm \sqrt{tr^2 - 4\,det}\right)$$

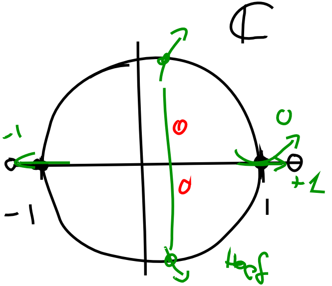
Jury conditions

Sometimes: $|\det J| < 1$

$\cancel{2} + \cancel{3}\det J > 0$

$$
\begin{aligned}
1 - \mathrm{tr}J + \det J &> 0, \quad \leftarrow \lambda < 1 \\
1 + \mathrm{tr}J + \det J &> 0, \quad \leftarrow \lambda > -1 \\
\to \quad 1 - \det J &> 0. \quad \leftarrow |\lambda| < 1
\end{aligned}
$$

$\}$ if $\lambda \in \mathbb{R}$



$\lambda^t = (e^{i\varphi})^t$ Neimark Sacker

If λ is an eigenvalue of J, then the stability boundary:

$\to$ $\lambda = 1$, for stability $\lambda < 1$

$\to$ $\lambda = -1$ $\qquad$ $\lambda > -1$

$\to$ $|\lambda| = 1$ but $\lambda \notin \mathbb{R}$ $\qquad$ $|\lambda| < 1$

$\hookrightarrow$ $\lambda = e^{i\phi}$

→ caterpillars

A host-parasitoid model: Nicholson–Bailey (1935)

↳ flies/wasps

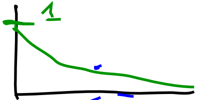
prob. of being found

$$N_{t+1} = e^r N_t e^{-\rho P_t}, \qquad P_{t+1} = N_t \left(1 - e^{\rho P_t}\right)$$

reproduce

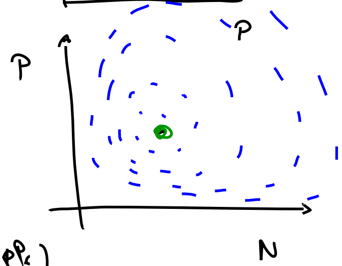
hosts

prob. of not being found



1) There is a positive steady state

2) it is unstable



- Does/can space stabilize?
- Use density dependent reproduction?

$$N_{t+1} = N e^{r(1-N_t)} e^{-\rho P_t}$$

$$P_{t+1} = N_t\left(1 - e^{-\rho P_t}\right) \simeq N_t \rho P_t$$

$$e^x = \underbrace{1 + x} + \dots$$

A simpler model

$$N_{t+1} = N_t e^{r(1-N_t-P_t)}, \qquad P_{t+1} = \rho N_t P_t$$

1) (0,0) trivial state

$$J = \begin{bmatrix} e^r & 0 \\ 0 & 0 \end{bmatrix}$$ unstable if $r > 0$

2) (N,0) semi-trivial state (1,0)

$$J = \begin{bmatrix} 1-r & \cancel{x} \\ 0 & \rho \end{bmatrix}$$ unstable if $r > 2 \to N$ alone

$\rho > 1 \to P$ invades

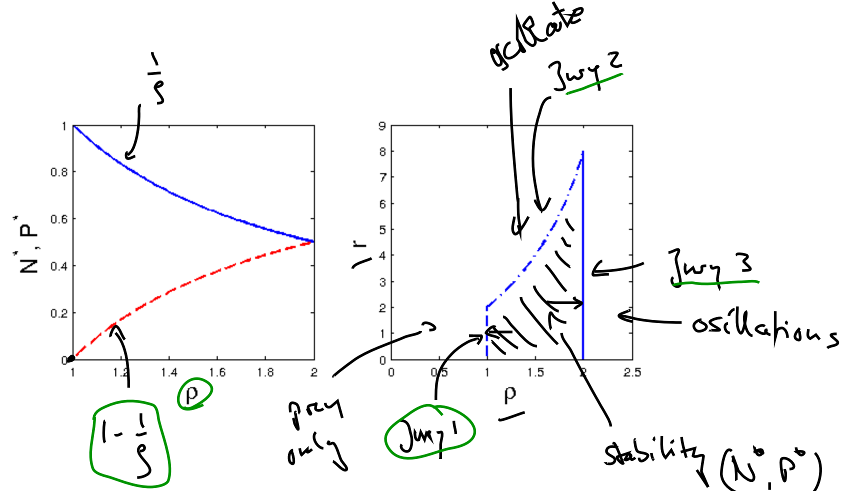
3) (N^*, P^*) coexistence state > 0

$\left(\frac{1}{\rho}, 1 - \frac{1}{\rho}\right)$

$$J = \begin{bmatrix} 1 - \frac{r}{\rho} & -\frac{r}{\rho} \\ \rho - 1 & 1 \end{bmatrix}$$

Stability of the steady state

Neubert & Kot (1992?)



How does space affect these dynamics?

The spatial model

dispersal

reproduction

host $\rightarrow$ $$N_{t+1}(x) = \int_\Omega K_N(x,y)F(N_t(y),P_t(y))dy$$

$$P_{t+1}(x) = \int_\Omega K_P(x,y)G(N_t(y),P_t(y))dy$$

- order of events
- insects: forest tent caterpillar (Cobbold et al. 2005)
- steady states
- stability
- effects of space and dispersal

The extinction state $(0, 0)$

$\mathcal{R} = \frac{\partial F}{\partial N}(0,0)$

$$\lambda\phi(x) = R\int_\Omega K_N(x,y)\phi(y)dy, \rightsquigarrow \text{know !}$$

$$\cancel{\lambda\psi(x)} = 0,$$

$\Rightarrow$ there is a cps

With Laplace kernel:

$$\boxed{L_N^* = \frac{2}{a_N\sqrt{R-1}}\arctan\left(\frac{1}{\sqrt{R-1}}\right).}$$

The prey-only state $(N^*(x), 0)$

$L\phi$

$(\lambda - \int k \dots)$

$$\begin{cases} \lambda\phi(x) = \int_\Omega K_N(x,y)[F_N(N^*(y),0)\phi(y) + \overbrace{F_P(N^*(y),0)\psi(y)}^{g}]dy \\ \lambda\psi(x) = \int_\Omega K_P(x,y)\underline{G_P(N^*(y),0)}\psi(y)dy \end{cases}$$

1) Second eq. decouples. Can solve separately.

2) Fredholm Alternative gives solution to the first eq.

$Lf = g$ is solvable if $\langle g, v\rangle = 0$ $\forall v \in \ker L^*$

if λ is not an eigenvalue of

$$\lambda\phi = \int K_N F_N \phi \, dy$$

Assume: $(N^*, 0)$ is stable w.r.t. perturbations $(\phi, 0)$

Fredholm alternative

$$\lambda\phi = \mathcal{K}\phi + f$$

Predator persistence

$\rho N^*(y)$

$$\lambda\psi(x) = \int_\Omega K_P(x,y) G_P(N^*(y),0)\psi(y)dy$$

- positive, compact operator
- dominant eigenvalue
- critical domain size $L_P^* > L_N^*$
- $G_P(N^*,0) = \rho N^*$
- Bounds for λ from using $\min N^* \leq N^* \leq \max N^*$
- Exact value for some $\bar{N}$

$\lambda_{min}\psi = \rho \min N \int K \psi dy$

$\bar{N}$ = avg. dis. sup.

$$\lambda\psi = \int \rho \cdot \bar{N} \cdot K_p(x,y)\psi(y)dy$$

Average dispersal success for the prey

$$\bar N_{ap} = \bar S_N \cdot F(\bar N_{ap}) \quad \Rightarrow \quad \boxed{\bar N_{\mathrm{ap}} = 1 + \frac{\ln \bar S_N}{r}}$$

$$\lambda \psi(x) = \rho \bar N_{ap} \int K_p(x-y)\,\psi(y)\,dy$$

$$\downarrow$$

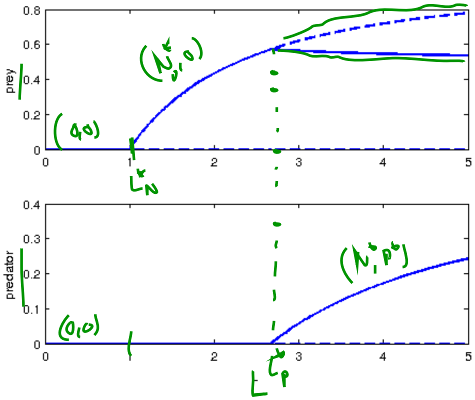
Laplace

$$\Downarrow$$

$$L_P^* \approx \frac{2}{a_P\sqrt{\rho \bar N_{\mathrm{ap}} - 1}} \arctan\left(\frac{1}{\sqrt{\rho \bar N_{\mathrm{ap}} - 1}}\right).$$

Illustration

based on av. disp. success.



Gobbeld et al 2005

Auto

The coexistence state $(N^*(x), P^*(x)) > 0$

$$\lambda\phi(x) = \int_\Omega K_N(x,y)[F_N^*\phi(y) + F_P^*\psi(y)]dy$$
$$\lambda\psi(x) = \int_\Omega K_P(x,y)[G_N^*\phi(y) + G_P^*\psi(y)]dy$$

- not necessarily positive
- not necessarily dominant eigenvalue
- possible: instability with oscillating eigenfunction

$(\bar{N}_{ap}, \bar{P}_{ap})$

$F_N^* = \frac{\partial F}{\partial N}(N^*, P^*)$

Case 1:

eigenfunction of one sign

Case 2:

eigenfunction changes sign $\rightarrow$ neut.

Average Dispersal Success