Integrodifference equations in spatial ecology

Lecture 11: Approximations for spread

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Approximations

- simplify analysis
- simplify computations
- accommodate data-poor situations
- prioritize data collection

The speed of spread

First c, then $N_t(x)$

for Kraecs eq.

$$N_{t+1}(x) = R\int_{-\infty}^{\infty} K(x-y)N_t(y)dy \qquad N_0(x) = \delta(x)$$

$K(z)$ is Gaussian with mean zero

$$K(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}}$$

$$\boxed{c_G^* = \sqrt{2\sigma^2 \ln(R)}.}$$

$$N_t = R^t K^{*t}$$

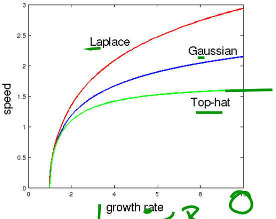
- Asymptotic quantity
- Convergence of convolutions to Gaussian
- Convergence of speed to Gaussian speed?

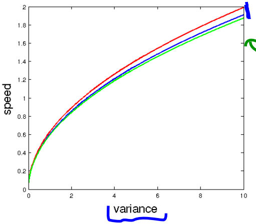
No – and a bit yes

Same variance

Same R

$c_{Laplace} > \boxed{c_G} > c_{TH}$

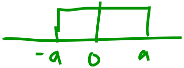




$\hat{c}$ R

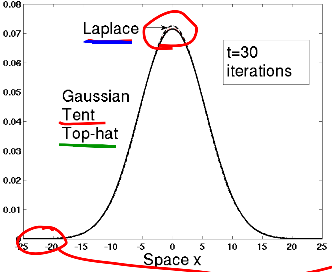
Cannot possibly be true (bounded support kernels)
Looks reasonably close to look for approximations

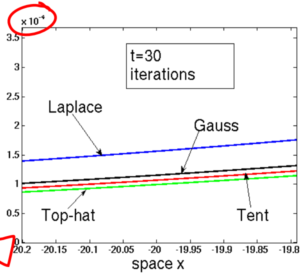
Top-hat Max per generation distance

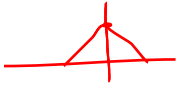


Rate of convergence – numerics

Same variance







K^{*30}

Convergence is slow

Rate of convergence – analytics

general $\quad N_t(x) \quad$ Gaussian

$$
\begin{aligned}
&\sup_x \left| R^t K^{*t}(x) - R^t G^{*t}(x; \mu, \sigma^2) \right| \\
= \;& R^t \sup_x \left| K^{*t}(x) - G(x; t\mu, t\sigma^2) \right| \\
= \;& \frac{R^t}{\sigma\sqrt{t}} \sup_y \left| \sigma\sqrt{t}\, K^{*t}(\underbrace{\sigma\sqrt{t}\, y + t\mu}_{=x}) - G(y; 0, 1) \right| \\
\sim \;& \frac{R^t}{\sigma\sqrt{t}} \frac{\text{const.}}{\sigma\sqrt{t}} = \frac{R^t}{\sigma^2 t} \text{const.}
\end{aligned}
$$

rescale

local limit theorem

$\frac{R^t}{t}$ as $t \to \infty$

Large deviations

- CLT is good on $[-\delta\sqrt{t}, \delta\sqrt{t}]$, $\delta > 0$.
- CLT fails outside $[-\delta t, \delta t]$, $\delta > 0$.
- Cramér's theorem

ct

$$\boxed{\int_{\mathbb{R}\setminus[-\delta t,\delta t]} K^{*t} dx} \sim e^{-\mu|t|}$$

Expansion I

Start with

$$\boxed{c(s) = \frac{1}{s}\ln(RM(s))}$$

$K(x) = K(-x)$

$M(s) = \int K(x) e^{sx}\, dx$

Expand

$$M(s) = 1 + \underbrace{\frac{\sigma^2}{2}s^2 + \frac{\mu_4}{24}s^4}_{x-1}, \quad \text{with} \quad \mu_4 = \int_{-\infty}^{\infty} x^4 K(x)dx$$

Now for the logarithm:

$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \ldots$

$\ln(RM(s)) = \ln R + \ln(M(s)) = \ln R + \frac{\sigma^2}{2}s^2 + \frac{\mu_4}{24}s^4 - \frac{1}{2}\left(\frac{\sigma^2}{2}s^2 + \frac{\mu_4}{24}s^4\right)^2$

$= \ln R + \frac{\sigma^2 s^2}{2} + \left(\frac{\mu_4}{24} - \frac{\sigma^4}{8}\right)s^4 + \ldots$

$= \ln R + \frac{\sigma^2 s^2}{2} + \frac{\mu_4/\sigma^4 - 3}{24} \cdot \sigma^4 s^4$

Expansion II

$$\ln(RM(s)) = \underbrace{\ln(R) + \frac{\sigma^2}{2}s^2 + \left(\frac{\mu_4}{24} - \frac{\sigma^4}{8}\right)s^4} = \ln(R) + \frac{\sigma^2}{2}s^2 + \epsilon\sigma^4 s^4,$$

where

$$\epsilon = \gamma_2/24 = \underline{(\mu_4/\sigma^4 - 3)/24}.$$

Excess kurtosis

For Gaussian: $\gamma_2 = 0$

For Laplace: $\gamma_2 = 3$

So ε is small.

Recall: $c(s) = \frac{1}{s}\ln(RM(s))$

Expansion III

$$s^2 = \frac{2\ln R}{\sigma^2} - (\ln R)^2 \varepsilon$$

The speed formula

$$c(s) = \frac{\ln(R)}{s} + \frac{\sigma^2}{2}s + \epsilon\sigma^4 s^3$$

Differentiate and find the zero

$$c'(s) = -\frac{\ln R}{s^2} + \frac{\sigma^2}{2} + 3\varepsilon\sigma^4 s^2 \stackrel{!}{=} 0$$

$$3\varepsilon\sigma^4 s^4 + \frac{\sigma^2}{2}s^2 - \ln R = 0$$

$$s^2 = \frac{1}{6\varepsilon\sigma^4}\left[-\frac{\sigma^2}{2} \oplus \sqrt{\frac{\sigma^4}{4} + 12\varepsilon\sigma^4\ln R}\right]$$

when $\varepsilon \to 0$

$$\sqrt{\frac{\sigma^4}{4} + 12\varepsilon\sigma^4\ln R}$$

$$= \frac{\sigma^2}{2} + \frac{1}{2\sqrt{\frac{\sigma^4}{4}}}12\varepsilon\sigma^4\ln R$$

$$- \frac{1}{8}\frac{1}{\sqrt{\frac{\sigma^4}{4}}^3}144\varepsilon^2\sigma^8(\ln R)^2$$

$$\sqrt{x+\varepsilon} = \sqrt{x} + \frac{1}{2\sqrt{x}}\varepsilon - \frac{1}{8\sqrt{x}^3}\varepsilon^2$$

Expansion IV

$c(s) = \frac{\ln R}{s} + \frac{\sigma^2}{2} s - \underbrace{\varepsilon \sigma^4 s^3}$

$$\boxed{s^2 = \frac{1}{\sigma^2}\left[2\ln(R) - 24\epsilon \ln(R)^2\right]}$$

Next: s, $1/s$, s^3

$\frac{1}{s} = \sqrt{\frac{\sigma^2}{2\ln R}} + 6\varepsilon\sqrt{\frac{\sigma^2 \ln R}{2}}$, $s^3 = \frac{2\ln R}{\sigma^2}\sqrt{\frac{2\ln R}{\sigma^2}}$

$\vdots$

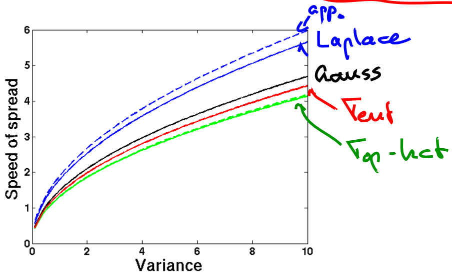
$c(s) = \dots = \underbrace{\sqrt{2\sigma^2 \ln R}}\left(1 + \varepsilon \cdot 2\ln R\right) = \hat{c}_G\left(1 + \frac{\gamma_2}{12}\ln R\right)$

Expansion V

Putting it all together

The kurtosis approximation

$$c(\gamma_2) = c_G\left(1 + \frac{\gamma_2}{12}\ln(R)\right)$$



$\gamma_2 = 0$ for Gauss

$\gamma_2 = 3$ for Laplace

$\gamma_2 < 0$ for Tent, Top-hat

Application

interpolates between Gaussian and Laplace.

The Gamma-Binomial kernel, variance $\boxed{\sigma^2}$ shape $\boxed{\alpha}$:

$$K_{GB}(x;\sigma,\alpha) = 2\frac{(\alpha/2)^{(\alpha/2+1/4)}}{\Gamma(\alpha)\sqrt{\pi}\sigma}\left|\frac{x}{\sigma}\right|^{\alpha-1/2} K_{1/2-\alpha}\left(\sqrt{2\alpha}\left|\frac{x}{\sigma}\right|\right),$$

where K_γ is the Bessel function of the second kind of order γ.

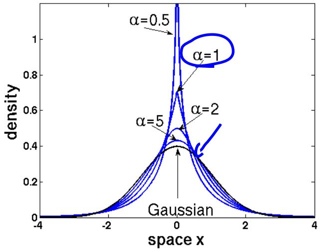
$$M(s) = \left(1 - \frac{\sigma^2 s^2}{2\alpha}\right)^{-\alpha} = \frac{1}{1 - \frac{\sigma^2}{2}s^2}$$

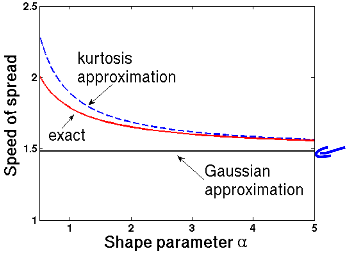
If $\alpha = 1$ then Laplace

excess kurtosis $\gamma_2 = 3/\alpha$

as $\alpha \to \infty$: $K_{GB} \to$ Gaussian

Kurtosis approximation





The shape of the solution

sol. $N_t = n_0 R^t \underbrace{K^{*t}}$

$$\boxed{N_{t+1} = R(K * N_t),} \qquad N_0 = n_0\delta(x)$$

$\downarrow$

Exponential transform

$$\tilde{N}(s) = \int N(x)e^{sx}dx$$

$\tilde{N}_t = n_0 R^t \cdot \tilde{K}^t(s)$

$\downarrow$

inverse transform?

$z = x - y$

$x = z + y$

convolution rule:

$$\widetilde{(K*N)}(s) = \iint K(x-y)N(y)\,dy\, e^{sx}\,dx$$

$$= \iint K(z)N(y)e^{s(z+y)}dy\,dz = \int K(z)e^{sz}dz \cdot \int N(y)e^{sy}dy$$

$$= \tilde{K} \cdot \tilde{N}$$

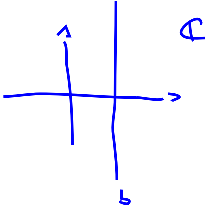
Inversion formula

Solution for exponential transform

$$\tilde{N}_t(s) = n_0 R^t \tilde{K}^t(s) = n_0 R^t M^t(s)$$

Inversion:

$$N_t(x) = \frac{n_0 R^t}{2\pi \mathrm{i}} \int_{b-\mathrm{i}\infty}^{b+\mathrm{i}\infty} M^t(s) e^{-sx} ds,$$



Choose b s.t. the integral converges.

The Gaussian kernel

$$M(s) = \exp(\sigma^2 s^2/2)$$

$$\int_{b-i\infty}^{b+i\infty} \exp\left(t\left(\frac{\sigma^2 s^2}{2} - \frac{x}{t}s\right)\right) ds$$

Expand around s_0, the minimum of the quadratic.

$$e^{-\frac{x^2}{2\sigma^2 t}} \int_{b-i\infty}^{b+i\infty} \exp\left(\frac{t\sigma^2}{2}(s-s_0)^2\right) ds$$

$(as-b)s$

$M(s) = e^{\frac{t\sigma^2 s^2}{2}}$

$a = \frac{\sigma^2 t}{2}, \; b = x$

$f(s) = as^2 - bs$

$= f(s_0) + c(s-s_0)^2$

s_0 = min of $f(s)$

$s_0 = \frac{b}{2a} = \frac{x}{\sigma^2 t}$

$f(s_0) = as_0^2 - bs_0$

$= \frac{\sigma^2 t}{2}\frac{x^2}{\sigma^4 t^2} - \frac{x^2}{\sigma^2 t}$

$= -\frac{x^2}{2\sigma^2 t}$

$e^{f(s_0) + c(s-s_0)^2}$

$e^{-\frac{x^2}{2\sigma^2 t}} \cdot e^{c(s-s_0)^2}$

indep of s

choose b

The Gaussian kernel

$s = s_0 + iy$

$s = s_0$ $\quad$ $s - s_0 = iy$

$ds = i\,dy$

Change variables

$$\boxed{\int_{s_0 - \mathrm{i}\infty}^{s_0 + \mathrm{i}\infty} e^{\frac{t\sigma^2}{2}(s - s_0)^2} ds} = \boxed{\mathrm{i} \int_{-\infty}^{+\infty} e^{-\frac{t\sigma^2}{2} y^2} dy} = \underbrace{\mathrm{i}\sqrt{\frac{2\pi}{\sigma^2 t}}}$$

$$\int e^{-\frac{x^2}{2\nu^2}} dx = \sqrt{2\pi\nu^2} \qquad \nu^2 = \frac{1}{t\sigma^2}$$

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Substitute back:

$$N_t(x) = n_0 R^t \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{x^2}{2\sigma^2 t}}$$

$$N_t(x) = \frac{n_0 R^t}{2\pi i} e^{-\frac{x^2}{2\sigma^2 t}} \, i\sqrt{\frac{2\pi}{\sigma^2 t}}$$

Now: $\tilde{N}_t = n_0 R^t M^t(s) = \frac{n_0 R^t}{2\pi i} \int_{b-i\infty}^{b+i\infty} M^t(s) e^{-sx} \, ds$

$\frac{\sigma^2 s^2}{t^2}$

$$e^{\ln M^t(s)} \cdot e^{-sx} = \exp\Big(\underbrace{t\big(\ln M(s) - s\tfrac{x}{t}\big)}_{f(s)}\Big)$$

$f(s) = f(s_0) + c(s-s_0)^2$

$\vdots$

$N_t(x) \approx \dots$

Saddle-Point approximation

General idea: expand the exponent in a quadratic.