Integrodifference equations in spatial ecology

Lecture 10: Dispersal success approximation

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Approximations

- simplify analysis
- simplify computations
- accommodate data-poor situations
- prioritize data collection

think:

$N_{t+1} = F(N_t)$ vs

$$N_{t+1}(x) = \int k(x,y) F(N_t(y))\,dy$$

Recall:

$$N_{t+1} = s F(N_t)$$

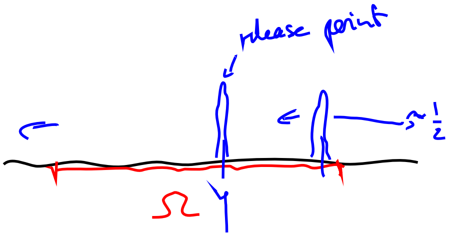
$K(x,y)$

empirically: $\forall y$: release many organisms at y

follow each of them.

s: prob. of staying in the patch

Point-release experiment



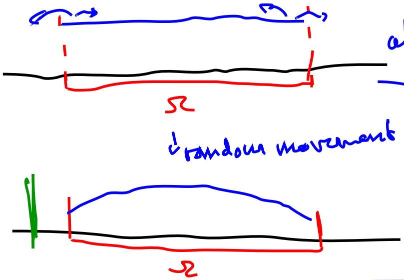
how many remain in Ω ?

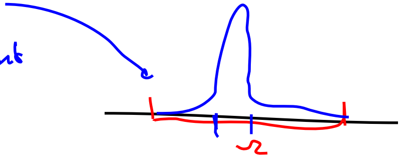
Dispersal success function $S(y) = \int_\Omega K(x, y)dx$ staying in Ω.

Average dispersal success: $\bar{S} = \frac{1}{|\Omega|} \int_\Omega S(y)dy$

Area release experiment

where do individuals end up after dispersal?





Redistribution function $U(x) = \int_\Omega K(x,y)dy$

NOT a probability, but $\bar{U} = \bar{S} \leq 1$.

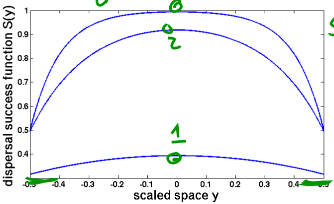
$$\bar{u} = \frac{1}{|\Omega|}\int_\Omega u(x)\,dx = \frac{1}{|\Omega|}\iint K(x,y)\,dx\,dy = \bar{S} \leq 1$$

Example: Laplace

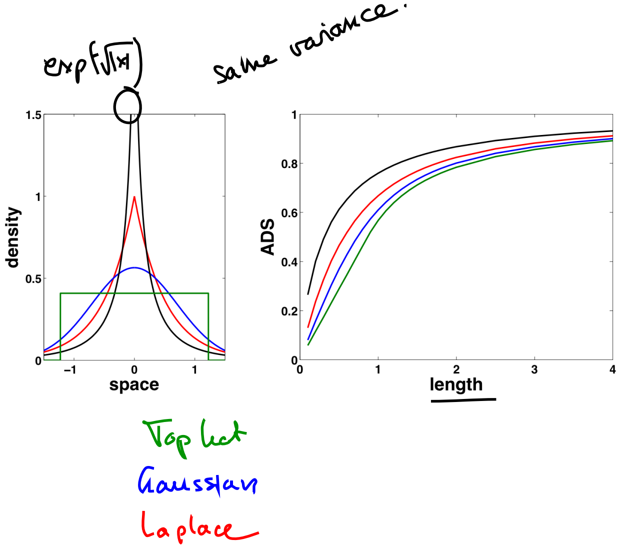
$$S(y) = U(y) = 1 - e^{-\hat{L}/2} \cosh(ay),$$

and the average dispersal success is

$$\bar{S} = 1 - \frac{1 - e^{-\hat{L}}}{\hat{L}},$$



Comparison of kernels



Dispersal success approximation of N^* (van Kirk & Lewis 1995/1997)

steady state

$$\longrightarrow \quad N^*(x) = \int_\Omega K(x,y)F(N^*(y))dy, \qquad \bar{N}^* = \frac{1}{|\Omega|}\int_\Omega N^*(x)dx$$

average

$$F(N^*(x)) = F(\bar{N}^*) + F'(\hat{N}(x))(N^*(x) - \bar{N}^*)$$

plug in

$$N^*(x) = \int K(x,y)\left[F(\bar{N}^*) + F'(\hat{N}(y))(N^*(y) - \bar{N}^*)\right]dy$$

$$= F(\bar{N}^*)\underbrace{\int K(x,y)\,dy}_{u(x)} + \int\underbrace{\int K(x,y)F'(\hat{N}(y))(N^*(y)-\bar{N}^*)\,dy}_{\text{small}}\,dx$$

$$N^*(x) \approx F(\bar{N}^*)u(x)$$

Integrate

$$\bar{N}^* = F(\bar{N}^*)\bar{s} + \frac{1}{|\Omega|}\int s(y)F'(\hat{N}(y))(N^*(y)-\bar{N}^*)\,dy$$

small & sloppy; small

Result:

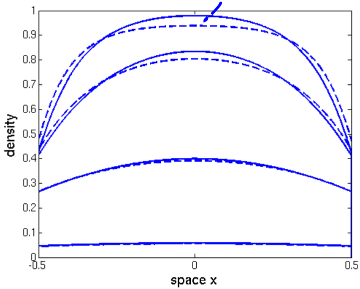
$$\bar{N}^* = \bar{s}F(\bar{N}^*)$$

$$N^*(x) \simeq F(\bar{N}^*)u(x)$$

Illustration

Steady state $N^*(x)$ vs $F(\bar{N}^*) \cdot U(x)$ with Laplace kernel

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \|$



$F(\bar{N}^*) S(x)$

Dispersal Success approximation.

Eigenvalue approximation

$$\lambda\phi(x) = \int_\Omega K(x,y)\phi(y)dy, \qquad \bar{\phi} = \frac{1}{|\Omega|}\int_\Omega \phi(x)dx = 1.$$

$$\lambda \phi(x) = \int K(x,y)[1 + \phi - 1]\,dy$$

$$= \int K(x,y)dy + \int K(x,y)(\phi(y) - 1)dy$$

$$= u(x) + \text{"small"}$$

$$\lambda \cdot 1 = \bar{s}$$

$$\Rightarrow \boxed{\lambda_{ap} = \bar{s}, \quad \phi_{ap} = \frac{u(x)}{\bar{s}}}$$

A conservative estimate

Lemma:

Assume that K is symmetric. Then $\lambda_{\mathrm{ap}} \leq \lambda$.

$\bar{S}$

⇒ if pop. can persist according to $\bar{S}$, then it can persist according to λ

Proof:

$$\lambda = \max_{\|u\|=1 \text{ in } L^2} \iint_\Omega u(x) K(x,y) u(y)\, dy\, dx \geq \frac{1}{|\Omega|} \iint K(x,y)\, dx\, dy = \bar{S}$$

choose $u(x) = \frac{1}{\sqrt{|\Omega|}}$

Improving the approximation: weighted average

$\lambda \phi = \int K(x,y)\phi(y)dy$

$\lambda \int \phi 1 \cdot dx = \iint K(x,y)\phi(y)\,dy\,dx = \int S(y)\phi(y)dy$

$\langle 1,\phi\rangle \qquad \langle S,\phi\rangle$

$\bar{S} = \frac{1}{|\Omega|}\iint K(x,y)\,dy\,dx \qquad \int S(y)\cdot 1\,dy = \langle S,1\rangle$

$|\Omega| = \int 1\cdot 1\,dx = \langle 1,1\rangle$

Notation: $\langle f,g\rangle = \int_\Omega f(x)g(x)dx$

Eigenvalue: $\lambda = \frac{\langle S,\phi\rangle}{\langle 1,\phi\rangle}$

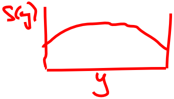
we have $\phi(x) \simeq U(x)$

Average: $\bar{S} = \frac{\langle S,1\rangle}{\langle 1,1\rangle}$

Weighted Average: $\lambda^{(2)}_{\text{ap}} = \frac{\langle S,U\rangle}{\langle 1,U\rangle} = \frac{\int_\Omega S(x)U(x)dx}{\int_\Omega U(x)dx}$

Reimer et al 2016: improve $\bar{S}$

average $\bar{S} = \frac{1}{|\Omega|}\int S(y)dy$ with uniform weight



Ideal: weight $= N^*(x)$, use $N^*(x) \sim U(x)$ $\quad \hat{S} := \frac{\int S(y)U(y)dy}{\int U(y)dy}$

Understanding the approximation

Use

$$\mathrm{cov}(S, U) = \frac{1}{|\Omega|}\int_\Omega (S(y) - \bar{S})(U(y) - \bar{S})dy = \frac{1}{|\Omega|}\int_\Omega S(y)U(y)dy - \bar{S}^2$$

and re-write

$$\lambda_{\mathrm{ap}}^{(2)} = \bar{S}\left(1 + \frac{1}{\bar{S}^2}\mathrm{cov}(S, U)\right).$$

λ_{ap}

$\lambda_{ap} = \bar{S} \leq \lambda$ if K is symmetric

if K is symmetric, then $\mathrm{cov}(S,U) = \mathrm{var}(S) > 0$

$\Rightarrow \lambda_{ap}^{(2)} > \lambda_{ap}$ $\rightarrow$ right direction.

Jane's challenge: show that $\lambda_{ap}^{(2)} \leq \lambda$ if K is symmetric!

The power method I

Goal: Given Matrix A, with pos. dom. eigenvalue λ, Find λ numerically.

Power method: define ϕ_n by ϕ_0 has norm 1.

$$\phi_{n+1} = \frac{A\phi_n}{\|A\phi_n\|} \quad \text{has norm } 1 \qquad \text{then} \quad \|A\phi_n\| \to \lambda \text{ as } n \to \infty, \quad \phi_n \to \phi$$

Apply these ideas to $\int K(x,y)\phi(y)dy$.

Start with $\phi_0(x) = \frac{1}{|\Omega|}$ const. then $\|\phi_0\|_{L^1} = 1$

$$\phi_1 = \frac{1}{\%}\int K(x,y)\frac{1}{|\Omega|}dy = \frac{1}{\%}\,\frac{1}{|\Omega|}\,u(x) = \frac{1}{\bar{s}|\Omega|}u(x)$$

$$\% = \int \frac{1}{|\Omega|}u(x)dx = \bar{s}$$

λ_{ap}, ϕ_{ap} are first step of power method.

The power method II

$$\phi_1 = \frac{1}{\int |\Omega|} u(x)\,dx$$

$$\phi_2 = \frac{1}{\%} \int k(x,y) \frac{1}{\int |\Omega|} u(y)\,dy$$ ⤳ need to know $k(x,y)$

$$\% = \iint k(x,y) \frac{1}{\int |\Omega|} u(y)\,dy\,dx = \int s(y)\, u(y)\,dy \frac{1}{\int |\Omega|} = \Lambda^{(2)}_{ap}$$

$$\lambda \phi = R \int k(x,y)\, \phi(y)\,dy$$

$R(y)$

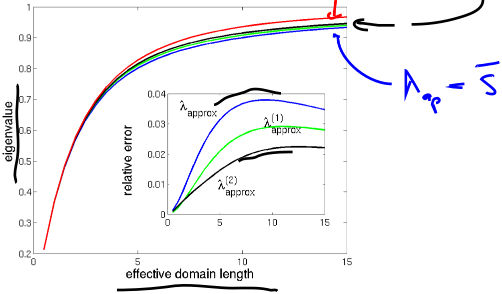
Eigenvalue .. in (2004)

$$\lambda \phi(x) = \int k(x,y)\, F'(N^*(y))\, \phi(y)\,dy$$

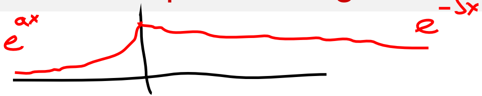
$N^*(y) \simeq \bar{N}^*$ How good is this?

The Laplace kernel

$$\lambda_{ap} = \bar{S}, \qquad \lambda_{ap}^{(2)} = \bar{S} + \mathrm{var}(S)$$

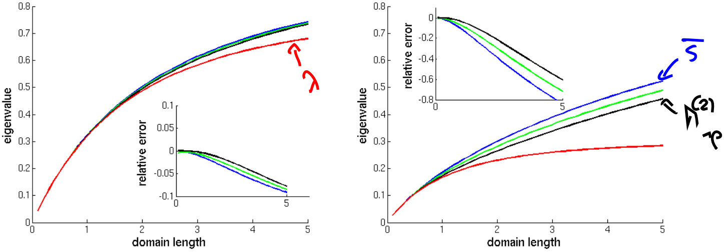


Asymmetric example: the eigenvalue



The asymmetric Laplace kernel

$$\lambda_{ap}^{(2)} = \bar{s}\left(1 + \frac{\mathrm{cov}(s,u)}{\bar{s}^2}\right)$$



if cov < 0

weak asymmetry

stronger asymmetry

Asymmetric example: the steady state